THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H University Mathematics 2017-2018
Suggested Solution to Assignment 3

1. (a)

$$\lim_{x\to 1} \frac{1-x}{2-\sqrt{x^2+3}} = \lim_{x\to 1} \frac{(1-x)(2+\sqrt{x^2+3})}{1-x^2} = \lim_{x\to 1} \frac{2+\sqrt{x^2+3}}{1+x} = 2.$$

(b)

$$\lim_{x\to\pi} \frac{\sin x}{\pi - x} = \lim_{x\to\pi} \frac{\sin(\pi - x)}{\pi - x} = 1.$$

(c)

$$\lim_{x\to 0} \frac{\sin 2x}{\tan 6x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{6x}{\sin 6x} \cdot \frac{\cos 6x}{3} = \frac{1}{3}.$$

(d)

$$\lim_{x\to+\infty} \sqrt{4x^2+x+1} - 2x = \lim_{x\to+\infty} \frac{4x^2+x+1-4x^2}{\sqrt{4x^2+x+1}+2x} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{\sqrt{4+\frac{1}{x}+\frac{1}{x^2}}+2} = \frac{1}{4}.$$

(e)

$$\lim_{x\to+\infty} (\frac{x+3}{x-2})^x = \lim_{x\to+\infty} \left((1+\frac{1}{\frac{x-2}{5}})^{\frac{x-2}{5}}\right)^{\frac{5x}{x-2}}$$

As $\lim_{x\to+\infty} \left((1+\frac{1}{\frac{x-2}{5}})^{\frac{x-2}{5}}\right. = e$ and $\lim_{x\to+\infty} \frac{5x}{x-2} = 5$,
so

$$\lim_{x\to+\infty} (\frac{x+3}{x-2})^x = e^5.$$

2. (a)

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \frac{2-x}{x^2-4} = \lim_{x\to 2-} \frac{-1}{x+2} = -\frac{1}{4}.$$

(b)

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} \frac{x-2}{x^2-4} = \lim_{x\to 2+} \frac{1}{x+2} = \frac{1}{4}.$$

(c)

$$\lim_{x\to -2} |x-2| = \lim_{x\to -2} (2-x) = 4.$$

$$\lim_{x\to -2} (x^2-4) = 0.$$

Therefore, $\lim_{x\to -2} f(x)$ does not exist.

3. (a) Because

$$-\sqrt{x-4} \le \sqrt{x-4} \cdot \cos\frac{1}{\sqrt{x-4}} \le \sqrt{x-4},$$

and

$$\lim_{x\to 4+} \pm\sqrt{x-4} = 0,$$

thus by sandwich theorem,

$$\lim_{x\to 4+} \sqrt{x-4} \cdot \cos\frac{1}{\sqrt{x-4}} = 0.$$

(b) Because

$$\frac{e^{-1}}{x} \le \frac{e^{\cos x}}{x} \le \frac{e}{x},$$

and $\lim_{x\to+\infty} \frac{e^{\pm}}{x} = 0$, thus

$$\lim_{x\to+\infty} \frac{e^{\cos x}}{x} = 0.$$

(c) Because

$$-1 - \tan 1 \le \cos(\tan x) - \tan(\cos x) \le 1 + \tan 1$$

, i.e. bounded,

$$\lim_{x\to+\infty} \frac{\cos(\tan x) - \tan(\cos x)}{2x+1} = 0$$

4. (a)

$$\frac{f(0)}{g(0)} = \frac{3}{4},$$

(b)

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f(x)}{x} \cdot \frac{\sin x}{g(x)} \cdot \frac{x}{\sin x} = 2 \times 1 \times 1 = 2.$$

(c)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x} \cdot x = 2 \times 0 = 0.$$

(d)

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{g(x)}{\sin x} \cdot \sin x = 1 \times 0 = 0.$$

5. (a) Because $\lim_{x\to 0^+} f(x) = e^0 - a = 1 - a$, $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0} f(x)$ exists,

$$1 - a = 1$$

i.e. $a = 0$

(b) Note that $f(0) = 1 = \lim_{x\to 0} f(x)$, this really mean $f(x)$ is continuous at $x = 0$.

6. proof.

By $0 \le |f(x)| \le |x-1|$, and $\lim_{x\to 1\pm} |x-1| = 0$.

by sandwich theorem,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 0 = f(0),$$

thus $f(x)$ is continuous at $x = 1$.

7. (a) Put $x = y = 0$ into the equation of the second condition, then we have $f(0) = [f(0)]^2$ and so $f(0) = 1$ or $f(0) = 0$ (rejected as $f(0) > 0$). Put $y = 0$ into the equation of the second condition, then for all real numbers x, we have

$$f(|x|) = f(\sqrt{x^2}) = f(x)f(0) = f(x).$$

(b) We are going to prove by using mathematical induction on n.

When $n = 1$, we have $f(\sqrt{1}x) = f(x) = [f(x)]^1$.

Assume that $f(\sqrt{n}x) = [f(x)]^n$ for some natural numbers n. Then we have

$$\begin{aligned} f(\sqrt{n+1}x) &= f(|\sqrt{n+1}x|) \\ &= f(\sqrt{n+1}|x|) \\ &= f(\sqrt{n+1}\cdot\sqrt{x^2}) \\ &= f(\sqrt{nx^2+x^2}) \\ &= f\left(\sqrt{(\sqrt{n}|x|)^2+|x|^2}\right) \\ &= f(\sqrt{n}|x|)f(|x|) \\ &= f(\sqrt{n}x)f(x) \\ &= [f(x)]^n f(x) \\ &= [f(x)]^{n+1} \end{aligned}$$

Therefore, by mathematical induction, $f(\sqrt{n}x) = [f(x)]^n$ for all natural numbers n.

(Remark: Be careful, when $x < 0$, $x \neq \sqrt{x^2}$ and that is the reason why we need the first equality.)

(c) We divide our discussion into three cases:

i. (r is a positive rational number) We let $r = \frac{p}{q}$ where p and q are some natural numbers. Use (b) by putting $x = \frac{1}{q}$ and $n = q^2$, we have $f(1) = [f(\frac{1}{q})]^{q^2}$ and so $[f(1)]^{1/q^2} = f(\frac{1}{q})$. Note that $f(\frac{p}{q}) = f(\sqrt{p^2}\cdot\frac{1}{q})$. Use (b) by putting $n = p^2$ and $x = \frac{1}{q}$, we have

$$f(\frac{p}{q}) = f(\sqrt{p^2}\cdot\frac{1}{q}) = [f(\frac{1}{q})]^{p^2} = [[f(1)]^{1/q^2}]^{p^2} = [f(1)]^{(p/q)^2}.$$

Therefore, we have $f(r) = [f(1)]^r$ for all positive rational numbers r.

You may have question that how we make a suitbale guess of x and n in the above.

Here is the idea:

We want to show that $f(\frac{p}{q}) = [f(1)]^{(p/q)^2}$. In order to see $f(\frac{p}{q})$, we have to use part (b) and put $x = \frac{1}{q}$ and $n = p^2$. Then, we have $f(\frac{p}{q}) = f(\sqrt{p^2}\cdot\frac{1}{q}) = [f(\frac{1}{q})]^{p^2}$.

Therefore, what remains to show is that $f(\frac{1}{q}) = [f(1)]^{1/q^2}$. We would like to prove it by using (b) again, but it would be difficult since the power n in the equation of (b) is a positive integer. To show the above, it is equivalent to show that $[f(1)] = [f(\frac{1}{q})]^{q^2}$. In order to see $f(1)$, we have to use part (b) and put $x = \frac{1}{q}$ and $n = q^2$ and the result follows.

ii. (r is a negative rational number) We have

$$f(r) = f(|r|) = [f(1)]^{|r|^2} = [f(1)]^{r^2}.$$

iii. ($r = 0$) Recall in (a) that we have $f(0) = 1$.

Therefore, $f(r) = [f(1)]^{r^2} = 1$ for the case $r = 0$.

Therefore, $f(r) = [f(1)]^{r^2}$ for all rational numbers r.

(d) Let a_n be a sequence of rational numbers such that $\lim_{n\to\infty} a_n = x$. Then we have

$$\begin{aligned}
f(a_n) &= [f(1)]^{a_n^2} \\
\lim_{n\to\infty} f(a_n) &= \lim_{n\to\infty} [f(1)]^{a_n^2} \\
f(\lim_{n\to\infty} a_n) &= [f(1)]^{\left(\lim_{n\to\infty} a_n\right)^2} \qquad (\because f\text{, exponential and square function are continuous functions}) \\
f(x) &= [f(1)]^{x^2}
\end{aligned}$$